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ORIGINAL PAPER

Bifurcation analysis of reaction–diffusion Schnakenberg model

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Abstract Bifurcations of spatially nonhomogeneous periodic orbits and steady state solutions are rigorously proved for a reaction–diffusion system modeling Schnakenberg chemical reaction. The existence of these patterned solutions shows the richness of the spatiotemporal dynamics such as oscillatory behavior and spatial patterns.

Keywords Schnakenberg model · Steady state solution · Hopf bifurcation · Steady state bifurcation · Pattern formation

Mathematics Subject Classification (2000) 58F07 · 58C28 · 58C15 · 34C23 · 35B20 · 35B32

1 Introduction

Schnakenberg model [13] is a basic differential equation model to describe an autocatalytic chemical reaction with possible oscillatory behavior. The trimolecular reactions between two chemical products X, Y and two chemical source A, B are in the following form:

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$$A \rightleftharpoons X,\ \ B \longrightarrow Y,\ \ 2X + Y \longrightarrow 3X. \tag{1.1}$$

Using the law of mass action, one can obtain a system of two reaction–diffusion equations for the concentrations $u(x, t)$ and $v(x, t)$ of the chemical products X and Y describing the reactions in (1.1). The non-dimensional form of the equations is

$$u_t - d_1 \Delta u = a - u + u^2 v, \quad v_t - d_2 \Delta v = b - u^2 v.$$

In the above equations d_1, d_2 are the diffusion coefficients of the chemicals X, Y, and a, b are the concentrations of A and B. It is also assumed that A and B are in abundance so a and b remain as constants. Because of its algebraic simplicity, Schnakenberg model has been used by many people as an exemplifying reaction–diffusion system to study spatiotemporal pattern formation [2,11,12].

In this paper we consider the reaction–diffusion Schnakenberg model in a bounded domain Ω:

$$\begin{cases} u_t - d_1 \Delta u = a - u + u^2 v, & x \in \Omega,\ t > 0, \\ v_t - d_2 \Delta v = b - u^2 v, & x \in \Omega,\ t > 0, \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^{\mathbb{N}}$, ($\mathbb{N} \geq 1$) is a bounded connected domain with a smooth boundary $\partial\Omega$; $u(x, t)$ and $v(x, t)$ are the concentrations of the chemical products X and Y at time t and location $x \in \Omega$; a no-flux boundary condition is assumed so that the chemical reactions are in a closed environment; a, b, d_1, d_2 are positive constants and the initial concentrations $u_0, v_0 \in C(\overline{\Omega})$ are non-negative functions. For simplicity, in this paper we only consider the case $\mathbb{N} = 1$ and $\Omega = (0, l\pi)$ for some $l > 0$.

In this paper, we analyze the stability of the unique positive steady state solution (u_*, v_*). When (u_*, v_*) loses its stability, we study the associated Hopf bifurcations and steady state bifurcations which give rise to temporal oscillations or non-constant stationary patterns. For simplicity of calculation, we use two new parameters:

$$b + a = \beta, \quad b - a = \alpha. \tag{1.3}$$

We fix the value of $\beta > 0$, and vary the value of α in $(-\beta, \beta)$. Our main results can be summarized as follows (a more detailed version is in Sect. 4):

1. When β is large, then the unique positive steady state (u_*, v_*) is locally asymptotically stable for any $\alpha \in (-\beta, \beta)$. Hence pattern formation is not likely.
2. If $0 < \beta < 1$ and d_2/d_1 is small, then (u_*, v_*) is locally asymptotically stable for $\alpha \in (-\beta, \beta^3)$, and it loses the stability at $\alpha = \beta^3$ through a Hopf bifurcation. In this case, a time-periodic pattern is likely to emerge for $\alpha \in (\beta^3, \beta)$.
3. If $0 < \beta < 1$ and d_2/d_1 is large, then there exists an $\alpha_j^S \in (0, \beta^3)$ such that (u_*, v_*) is locally asymptotically stable for $\alpha \in (-\beta, \alpha_j^S)$, and it loses the stability at $\alpha = \alpha_j^S$ through a steady state bifurcation. In this case, a stationary spatially non-homogenous pattern is likely to appear for $\alpha \in (\alpha_j^S, \beta)$.

In Sect. 2, we do a Hopf bifurcation analysis with parameter α, and in Sect. 3, we analyze the steady state solution bifurcations. Some numerical simulations and a summary of change of dynamics are given in Sect. 4. Our analysis follows the approach given in Yi et al. [22] for diffusive predator-prey systems, and similar approach has also been used in [3,6,9,16,19–21] for other reaction–diffusion models. Note that both patterns described above were first considered by Turing [15] in his seminal work 60 years ago.

There have been some previous work on reaction–diffusion Schnakenberg model. It has been shown in [5,17,18] that for small values of d_1 the system (1.2) exhibits various stationary multi-spot and spike patterns. Our work explains such multi-spike solutions from the view of bifurcation. Very recently the steady state solutions of (1.2) (or more general form) have also been considered in [2,8], and the Hopf bifurcations of (1.2) has been investigated in [19]. Our results here use a simpler parametrization (1.3) which simplifies the expression of many calculations. Our results not only sharpen many previous results, but also is the only one to provide a panoramic view of the dynamics of (1.2). We also mentioned that it is known that the limit cycle of the corresponding ODE system of (1.2) is unique (see [4]).

In the paper we use (u_α, v_α) to denote the unique constant steady state solution of (1.2). We denote by $\mathbb{N}$ the set of all the positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2 Hopf bifurcations

In this section, we consider the Hopf bifurcations for the Schnakenberg model. In particular we derive an explicit algorithm for direction of Hopf bifurcation and stability of the bifurcating periodic solutions for the reaction–diffusion system of Schnakenberg model with Neumann boundary condition. While our calculations can be carried over to higher spatial domains, we restrict ourselves to the case of one-dimensional spatial domain $(0, l\pi)$, for which the structure of the eigenvalues is clear. The Schnakenberg model in the spatial domain $\Omega = (0, l\pi)$ is in form

$$\begin{cases} u_t - d_1 u_{xx} = a - u + u^2 v, & x \in (0, l\pi),\ t > 0, \\ v_t - d_2 v_{xx} = b - u^2 v, & x \in (0, l\pi),\ t > 0, \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x), & x \in (0, l\pi), \\ u_x(0, t) = u_x(l\pi, t) = v_x(0, t) = v_x(l\pi, t) = 0, & t > 0. \end{cases} \tag{2.1}$$

First we consider the corresponding kinetic system of (2.1).

$$\begin{cases} u' = a - u + u^2 v, & t > 0, \\ v' = b - u^2 v, & t > 0, \\ u(0) = u_0 > 0,\ v(0) = v_0 > 0. \end{cases} \tag{2.2}$$

An equilibrium point (u, v) of (2.2) satisfies

$$\begin{cases} a - u + u^2 v = 0, \\ b - u^2 v = 0. \end{cases}$$

By a simple calculation, $\left(a+b, \dfrac{b}{(a+b)^2}\right)$ is the unique positive equilibrium point of (2.2). Let

$$a+b=\beta, \quad b-a=\alpha.$$

Then $a = \dfrac{\beta-\alpha}{2}, b = \dfrac{\beta+\alpha}{2}$, and $a > 0, b > 0$ is equivalent to $\beta > 0$ and $\beta > \alpha > -\beta$. We shall rewrite the equilibrium point by $(\beta, \dfrac{\beta+\alpha}{2\beta^2}) \equiv (u_\alpha, v_\alpha)$. In the following we choose a fixed parameter $\beta > 0$, and use α as the main bifurcation parameter. The Jacobian matrix of system (2.2) evaluated at $(\beta, \dfrac{\beta+\alpha}{2\beta^2})$ is

$$L_0(\alpha) = \begin{pmatrix} \dfrac{\alpha}{\beta} & \beta^2 \\ -1-\dfrac{\alpha}{\beta} & -\beta^2 \end{pmatrix}. \tag{2.3}$$

The characteristic equation of $L_0(\alpha)$ is

$$\mu^2 - T(\alpha)\mu + D(\alpha) = 0, \tag{2.4}$$

where $T(\alpha) = -\beta^2 + \dfrac{\alpha}{\beta}$, $D(\alpha) = \beta^2$. The eigenvalues $\mu(\alpha)$ of $L_0(\alpha)$ are given by

$$\mu(\alpha) = \frac{T(\alpha) \pm \sqrt{T^2(\alpha) - 4D(\alpha)}}{2}.$$

A Hopf bifurcation value α satisfies the following condition:

$$T(\alpha) = 0, \quad D(\alpha) > 0, \quad \text{and} \quad T'(\alpha) \neq 0.$$

Clearly we always have $D(\alpha) > 0$, and $T(\alpha) = 0$ implies $\alpha = \beta^3$. Finally $T'(\alpha) = 1/\beta$ for any $\beta > 0$. Then $\alpha = \beta^3$ is the only Hopf bifurcation point for (2.2). Since we assume that $\beta > \alpha$, hence when $0 < \beta < 1$, there exists a unique Hopf bifurcation point $\alpha = \beta^3$. When $\beta \geq 1$ and $\beta > \alpha$, the unique equilibrium point of (2.2) is always locally asymptotically stable.

Next we consider Hopf bifurcations from the constant steady state (u_α, v_α) of the reaction–diffusion system (here we use equivalent parameters α and β)

$$\begin{cases} u_t = d_1 u_{xx} + \dfrac{\beta-\alpha}{2} - u + u^2 v, & x \in (0, l\pi),\ t > 0, \\ v_t = d_2 v_{xx} + \dfrac{\beta+\alpha}{2} - u^2 v, & x \in (0, l\pi),\ t > 0, \\ u_x(x,t) = v_x(x,t) = 0, & x = 0, l\pi,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in (0, l\pi). \end{cases} \tag{2.5}$$

Again (2.5) has a unique positive constant steady state $(u_\alpha, v_\alpha) = (\beta, \frac{\alpha+\beta}{2\beta^2})$. We choose the fixed parameters β, l properly, and use α as the main bifurcation parameter. Define

$$X = \{(u, v) \in H^2[(0, l\pi)] \times H^2[(0, l\pi)] | \, u'(0) = v'(0) = u'(l\pi) = v'(l\pi) = 0\}.$$

The linearized operator of system (2.5) at $(\beta, \frac{\alpha+\beta}{2\beta^2})$ is

$$L(\alpha) = \begin{pmatrix} d_1 \frac{\partial^2}{\partial x^2} + \frac{\alpha}{\beta} & \beta^2 \\ -1 - \frac{\alpha}{\beta} & d_2 \frac{\partial^2}{\partial x^2} - \beta^2 \end{pmatrix}.$$

It is well-known that eigenvalue problems

$$-\psi'' = \mu\psi, \; x \in (0, l\pi), \; \psi'(0) = \psi'(l\pi) = 0,$$

has eigenvalues $\mu_n = n^2/l^2 (n = 0, 1, 2, \ldots)$, with corresponding eigenfunctions $\psi_n(x) = \cos \frac{n}{l} x$. Let

$$\begin{pmatrix} \phi \\ \varphi \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{n}{l} x \tag{2.6}$$

be an eigenfunction for $L(\alpha)$ with eigenvalue $\mu(\alpha)$, that is, $L(\alpha)(\phi, \varphi)^T = \mu(\alpha)(\phi, \varphi)^T$. From [22], there exists $n \in \mathbb{N}_0$ such that $L_n(\alpha)(a_n, b_n)^T = \mu(\alpha)(a_n, b_n)^T$, where L_n is define by

$$L_n(\alpha) = \begin{pmatrix} -\frac{d_1 n^2}{l^2} + \frac{\alpha}{\beta} & \beta^2 \\ -1 - \frac{\alpha}{\beta} & -\frac{d_2 n^2}{l^2} - \beta^2 \end{pmatrix}. \tag{2.7}$$

The characteristic equation of $L_n(\alpha)$ is

$$\mu^2 - T_n(\alpha)\mu + D_n(\alpha) = 0, \quad n = 0, 1, 2, \ldots, \tag{2.8}$$

where

$$T_n(\alpha) := \frac{\alpha}{\beta} - \beta^2 - \frac{n^2}{l^2}(d_1 + d_2), \tag{2.9}$$

$$D_n(\alpha) := d_1 d_2 \frac{n^4}{l^4} + \left(d_1\beta^2 - \frac{\alpha d_2}{\beta} \right) \frac{n^2}{l^2} + \beta^2. \tag{2.10}$$

and the eigenvalues $\mu(\alpha)$ of $L_n(\alpha)$ are given by

$$\mu(\alpha) = \frac{T_n(\alpha) \pm \sqrt{T_n^2(\alpha) - 4D_n(\alpha)}}{2}, \quad n = 0, 1, 2, \ldots.$$

We identify the Hopf bifurcation value α satisfying the following condition.

(H_1) There exists $n \in \mathbb{N}_0$, such that

$$T_n(\alpha) = 0,\ D_n(\alpha) > 0,\ T_j(\alpha) \neq 0,\ D_j(\alpha) \neq 0, \quad \text{for any } j \neq n.$$

Let the unique pair of complex eigenvalues near the imaginary axis be $\gamma(\alpha) \pm i\omega(\alpha)$, then the following transversality condition holds:

$$\gamma'(\alpha) \neq 0. \tag{2.11}$$

For $j \in \mathbb{N}_0$, define

$$\alpha_j^H = \beta^3 + \beta\frac{j^2}{l^2}(d_1 + d_2), \tag{2.12}$$

then $T_j(\alpha_j^H) = 0$ and $T_i(\alpha_j^H) \neq 0$ for $i \neq j$. Define

$$l_n = n\sqrt{\frac{d_1 + d_2}{1 - \beta^2}}, \qquad n \in \mathbb{N}_0. \tag{2.13}$$

Then for $l_n < l \leq l_{n+1}$, we have exactly $n + 1$ possible Hopf bifurcation points $\alpha = \alpha_j^H$ $(0 \leq j \leq n)$ defined by (2.12), and these points satisfy that

$$\alpha_0^H (=\beta^3) < \alpha_1^H < \cdots < \alpha_n^H < \beta.$$

Next we verify whether $D_i(\alpha_j^H) \neq 0$ for all $i \in \mathbb{N}_0$, and in particular, $D_j(\alpha_j^H) > 0$. We claim that if

$$(\sqrt{2} - 1)\sqrt{\frac{d_2}{d_1}} < \beta < 1, \tag{2.14}$$

then $D_i(\alpha_j^H) > 0$ for any $i \in \mathbb{N}_0$ and $\alpha_j^H \in (0, \beta)$. Indeed since $\alpha_j^H/\beta < 1$, then from (2.14),

$$\begin{aligned} D_i(\alpha_j^H) &= p_i^2 d_1 d_2 + p_i\left(d_1\beta^2 - \frac{\alpha_j^H d_2}{\beta}\right) + \beta^2 \\ &> p_i^2 d_1 d_2 + p_i(d_1\beta^2 - d_2) + \beta^2, \end{aligned} \tag{2.15}$$

where $p_i = \frac{i^2}{l^2}$. If $\beta \geq \sqrt{d_2/d_1}$, then $d_1\beta^2 - d_2 \geq 0$ and $D_i(\alpha_j^H) > 0$ from (2.15). If

$$(\sqrt{2}-1)\sqrt{\frac{d_2}{d_1}} < \beta < \sqrt{\frac{d_2}{d_1}},$$

then from (2.15),

$$\begin{aligned} D_i(\alpha_j^H) &> p_i^2 d_1 d_2 + p_i(d_1\beta^2 - d_2) + \beta^2 \geq \beta^2 - \frac{d_1 d_2}{4}\left(\frac{\beta^2}{d_2} - \frac{1}{d_1}\right)^2 \\ &= -\frac{d_1}{4d_2}\left(\beta^2 - (\sqrt{2}+1)^2\frac{d_2}{d_1}\right)\left(\beta^2 - (\sqrt{2}-1)^2\frac{d_2}{d_1}\right) > 0. \end{aligned}$$

Finally let the eigenvalues close to the pure imaginary one near at $\alpha = \alpha_j^H$ be $\gamma(\alpha) \pm i\omega(\alpha)$. Then

$$\gamma'(\alpha_j^H) = \frac{T_j'(\alpha_j^H)}{2} = \frac{1}{\beta} > 0.$$

Now by using the Hopf bifurcation theorem in [22], we have

Theorem 2.1 *Let l_n as defined in (2.13), and assume that $l_n < l \leq l_{n+1}$ for some $n \in \mathbb{N}_0$. Suppose that d_1, d_2 and β satisfy (2.14). Then for (2.5), there exist $n+1$ Hopf bifurcation points α_j^H $(0 \leq j \leq n)$ defined by (2.12), satisfying*

$$\beta^3 = \alpha_0^H < \alpha_1^H < \alpha_2^H < \cdots < \alpha_n^H < \beta.$$

At each $\alpha = \alpha_j^H$, the system (2.5) undergoes a Hopf bifurcation, and the bifurcating periodic orbits near $(\alpha, u, v) = \left(\alpha_j^H, \beta, \frac{\alpha_j^H + \beta}{2\beta^2}\right)$ can be parameterized as $(\alpha(s), u(s), v(s))$, so that $\alpha(s) \in C^\infty$ in the form of $\alpha(s) = \alpha_j^H + o(s)$, $s \in (0, \delta)$ for some small $\delta > 0$, and

$$\begin{cases} u(s)(x,t) = \beta + s\left(a_n e^{2\pi i t/T(s)} + \overline{a_n} e^{-2\pi i t/T(s)}\right)\cos\frac{n}{l}x + o(s), \\ v(s)(x,t) = \dfrac{\alpha_j^H + \beta}{2\beta^2} + s\left(b_n e^{2\pi i t/T(s)} + \overline{b_n} e^{-2\pi i t/T(s)}\right)\cos\frac{n}{l}x + o(s), \end{cases} \tag{2.16}$$

where (a_n, b_n) is the corresponding eigenvector, and $T(s) = 2\pi/\sqrt{D_j(\alpha_j^H)} + o(s)$ (D_j is defined in (2.10)). Moreover

1. *The bifurcating periodic orbits from $\alpha = \alpha_0^H = \beta^3$ are spatially homogeneous, which coincide with the periodic orbits of the corresponding ODE system;*
2. *The bifurcating periodic orbits from $\alpha = \alpha_j^H$ are spatially nonhomogeneous.*

Next we follow the methods in [22] to calculate the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits bifurcating from $\alpha = \alpha_0^H$. We have the following result.

Theorem 2.2 *For equation* (2.5)*, when* $0 < \beta < 1$*, the Hopf bifurcation at* $\alpha_0^H = \beta^3$ *is supercritical. That is, for a small* $\varepsilon > 0$ *and* $\alpha \in (\alpha_0^H, \alpha_0^H + \varepsilon)$*, there is a small amplitude spatially homogenous periodic orbit, and this periodic orbit is locally asymptotically stable orbits.*

Proof Here we follow the notations and calculations in [22]. When $\alpha = \alpha_0^H = \beta^3$, Eq. (2.8) has a pair of purely imaginary eigenvalues $\mu = \pm i\beta$. For the Jacobian matrix

$$L_0(\alpha) = \begin{pmatrix} \beta^2 & \beta^2 \\ -1-\beta^2 & -\beta^2 \end{pmatrix}, \tag{2.17}$$

an eigenvector q of eigenvalue $i\beta$ satisfying

$$L_0 q = i\beta q$$

can be chosen as $q := (a_0, b_0)^T = (-\beta, \beta - i)^T$. Define the inner product in $X_{\mathbb{C}}$ by

$$\langle U_1, U_2 \rangle = \int_0^{l\pi} (\overline{u_1} u_2 + \overline{v_1} v_2) dx,$$

where $U_i = (u_i, v_i)^T \in X_{\mathbb{C}}^2$, $i = 1, 2$. We choose an associated eigenvector q^* for the eigenvalue $\mu = -i\beta$ satisfying

$$L_0^* q^* = -i\beta q^*, \quad \langle q^*, q \rangle = 1, \quad \langle q^*, \bar{q} \rangle = 0.$$

Then $q^* := (a_0^*, b_0^*)^T = \left(\dfrac{-1 - i\beta}{2\beta l\pi}, \dfrac{-i}{2l\pi} \right)^T$.

Let $f(u, v) = a - u + u^2 v$ and $g(u, v) = b - u^2 v$, by calculation, at $\left(\beta, \dfrac{\beta + \beta^3}{2\beta^2}\right)$, we have

$$\begin{cases} g_{uu} = -f_{uu}, \ g_{uv} = -f_{uv}, \ g_{uuv} = -f_{uuv}, \\ f_{vv} = f_{vvv} = f_{uvv} = f_{uuu} = g_{uuu} = g_{vv} = g_{uvv} = g_{vvv} = 0, \\ f_{uu} = \dfrac{1+\beta^2}{\beta}, \ f_{uv} = 2\beta, \ f_{uuv} = 2. \end{cases} \tag{2.18}$$

By direct calculation, it follows that

$$\begin{aligned} c_0 &= f_{uu} a_0^2 + 2 f_{uv} a_0 b_0 + f_{vv} b_0^2 = \beta - 3\beta^3 + 4\beta^2 i, \\ d_0 &= g_{uu} a_0^2 + 2 g_{uv} a_0 b_0 + g_{vv} b_0^2 = -c_0. \\ e_0 &= f_{uu} |a_0|^2 + f_{uv} (a_0 \overline{b_0} + \overline{a_0} b_0) + f_{vv} |b_0|^2 = \beta - 3\beta^3, \end{aligned}$$

$$f_0 = g_{uu}|a_0|^2 + g_{uv}(a_0\overline{b_0} + \overline{a_0}b_0) + g_{vv}|b_0|^2 = -e_0.$$
$$g_0 = f_{uuu}|a_0|^2 a_0 + f_{uuv}(2|a_0|^2 b_0 + a_0^2\overline{b_0}) + f_{uvv}(2|b_0|^2 a_0 + b_0^2\overline{a_0})a = 6\beta^3 - 2\beta^2 i,$$
$$h_0 = g_{uuu}|a_0|^2 a_0 + g_{uuv}(2|a_0|^2 b_0 + a_0^2\overline{b_0}) + g_{uvv}(2|b_0|^2 a_0 + b_0^2\overline{a_0}) = -g_0.$$

Denote

$$Q_{qq} = \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}, \quad Q_{q\bar{q}} = \begin{pmatrix} e_0 \\ f_0 \end{pmatrix}, \quad C_{qq\bar{q}} = \begin{pmatrix} g_0 \\ h_0 \end{pmatrix}. \tag{2.19}$$

Then

$$\langle q^*, Q_{qq}\rangle = \int_0^{l\pi} \left(\frac{-1+i\beta}{2\beta l\pi} c_0 + \frac{i}{2l\pi} d_0 \right) dx = -\frac{c_0}{2\beta},$$
$$\langle q^*, Q_{q\bar{q}}\rangle = \int_0^{l\pi} \left(\frac{-1+i\beta}{2\beta l\pi} e_0 + \frac{i}{2l\pi} f_0 \right) dx = -\frac{e_0}{2\beta},$$
$$\langle q^*, C_{qq\bar{q}}\rangle = \int_0^{l\pi} \left(\frac{-1+i\beta}{2\beta l\pi} g_0 + \frac{i}{2l\pi} h_0 \right) dx = -\frac{g_0}{2\beta},$$
$$\langle \bar{q}^*, Q_{qq}\rangle = \int_0^{l\pi} (-\beta c_0 + (\beta + i) d_0)\, dx = -\frac{c_0}{2\beta},$$
$$\langle \bar{q}^*, Q_{q\bar{q}}\rangle = \int_0^{l\pi} (-\beta e_0 + (\beta + i) f_0)\, dx = -\frac{e_0}{2\beta},$$
$$\langle q^*, Q_{qq}\rangle = \langle \bar{q}^*, Q_{qq}\rangle,$$
$$\langle q^*, C_{qq\bar{q}}\rangle = \langle \bar{q}^*, Q_{q\bar{q}}\rangle.$$

Hence

$$\begin{aligned} H_{20} &= (c_0, d_0)^T + \frac{c_0}{2\beta}(a_0, b_0)^T + \frac{c_0}{2\beta}(\overline{a_0}, \overline{b_0})^T \\ &= c_0(1, -1)^T + c_0(-1, 1)^T = 0, \\ H_{11} &= (e_0, f_0)^T + \frac{e_0}{2\beta}(a_0, b_0)^T + \frac{e_0}{2\beta}(\overline{a_0}, \overline{b_0})^T \\ &= e_0(1, -1)^T + e_0(-1, 1)^T = 0, \end{aligned}$$

which implies that $\omega_{20} = \omega_{11} = 0$, then

$$\langle q^*, Q_{\omega_{11}q}\rangle = \langle q^*, Q_{\omega_{20}\bar{q}}\rangle = 0. \tag{2.20}$$

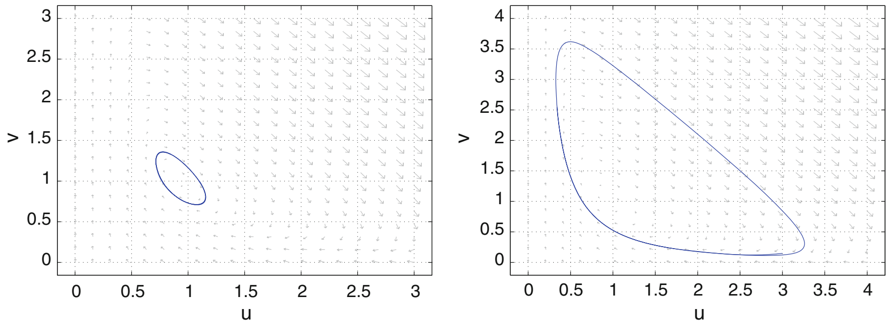

Fig. 1 Phase portraits and periodic orbits for the ODE system corresponding to (2.5) when $\beta = 0.9$. The *horizontal axis* is u, and the *vertical axis* is v. *Left*: $\alpha = 0.74$, a small amplitude limit cycle; *Right*: $\alpha = 0.88$, a large amplitude limit cycle. Here the Hopf bifurcation point $\alpha_0^H = 0.729$

Thus

$$\begin{aligned}\mathrm{Re}(c_1(\alpha)) &= Re\left\{\frac{i}{2\omega_0}\langle q^*, Q_{qq}\rangle \cdot \langle q^*, Q_{q\bar{q}}\rangle + \frac{1}{2}\langle q^*, C_{qq\bar{q}}\rangle\right\}\\ &= -\frac{1-3\beta^2}{2} - \frac{3\beta^2}{2} = -\frac{1}{2},\end{aligned}$$

Moreover $\mathrm{Re}(c_1(\alpha)) < 0$, since $T(\alpha) = -\beta^2 + \dfrac{\alpha}{\beta}$, hence $T'(\alpha) = \dfrac{1}{\beta} > 0$. Therefore when $\alpha > \alpha_0^H = \beta^3$, the equilibrium point of (2.2) is unstable, and the system must have a periodic orbit by the Poincaré–Bendixson theorem. From the result in [4], the periodic orbit must be unique. From calculation above, the Hopf bifurcation at $\alpha = \alpha_0^H$ is supercritical; and when $\alpha \in (\alpha_0^H, \alpha_0^H + \varepsilon)$, the bifurcating periodic orbit is locally asymptotically stable. □

The periodic orbits of system (2.2) for some parameters are shown in Fig. 1, and a bifurcation diagram of the periodic orbits is shown in Fig. 2.

3 Steady state bifurcation

In this section, we consider the steady state bifurcations of the system (2.1). We consider the system of two coupled ordinary differential equations:

$$\begin{cases} d_1 u'' + \dfrac{\beta-\alpha}{2} - u + u^2 v = 0, & x \in (0, l\pi),\\ d_2 v'' + \dfrac{\beta+\alpha}{2} - u^2 v = 0, & x \in (0, l\pi),\\ u'(0) = u'(l\pi) = v'(0) = v'(l\pi) = 0, \end{cases} \tag{3.1}$$

which is a special case of more general steady state equation for a higher dimensional domain:

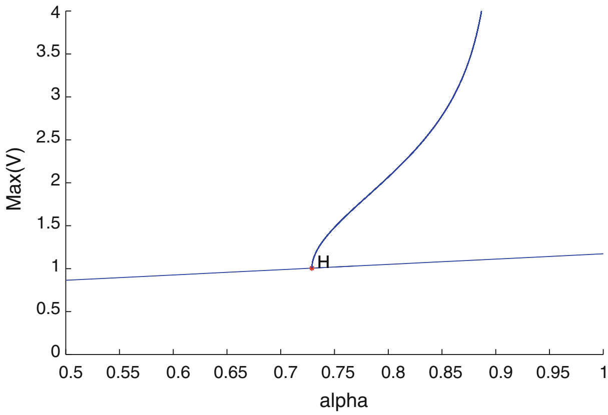

Fig. 2 Bifurcation of periodic orbits for the ODE system corresponding to (2.5) when $\beta = 0.9$. The *horizontal axis* is α, and the *vertical axis* is max $v(t)$ for the limit cycle $(u(t), v(t))$. Here the Hopf bifurcation point $\alpha_0^H = 0.729$

$$\begin{cases} -d_1 \Delta u = \dfrac{\beta-\alpha}{2} - u + u^2 v, & x \in \Omega, \\ -d_2 \Delta v = \dfrac{\beta+\alpha}{2} - u^2 v, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

The following a priori estimate can be easily established via maximum principle, and we omit the proof (see [2,8]).

Lemma 3.1 *Suppose that* $\beta > 0$ *and* $\beta > \alpha > -\beta$. *Then any positive solution* (u, v) *of* (3.2) *satisfies*

$$\frac{\beta-\alpha}{2} \le u(x) \le \beta + \frac{2d_2(\beta+\alpha)}{d_1(\beta-\alpha)^2}, \qquad x \in \overline{\Omega}, \tag{3.3}$$

and

$$\frac{\alpha+\beta}{2\left[\beta + \dfrac{2d_2(\beta+\alpha)}{d_1(\beta-\alpha)^2}\right]^2} \le v(x) \le \frac{2(\beta+\alpha)}{(\beta-\alpha)^2}, \qquad x \in \overline{\Omega}. \tag{3.4}$$

Recall $D_n(\alpha)$ and $T_n(\alpha)$ defined in (2.9) and (2.10). Now we identify steady state bifurcation values α of steady state system (3.1), which satisfy the following condition:

(H_2) there exists $n \in \mathbb{N}_0$ such that

$$D_n(\alpha) = 0, \; T_n(\alpha) \neq 0, \; D_j(\alpha) \neq 0, \; T_j(\alpha) \neq 0, \quad \text{for any } j \neq n; \tag{3.5}$$

and

$$\frac{d}{d\alpha} D_n(\alpha) \neq 0. \tag{3.6}$$

Apparently, $D_0(\alpha) = \beta^2 \neq 0$ for $\alpha > 0$, hence we only consider $n \in \mathbb{N}$. In the following we fix an arbitrary $\beta > 0$, to determine α values satisfying condition (H_2), we notice that $D_n(\alpha) = 0$ is equivalent to

$$D(\alpha, p) = d_1 d_2 p^2 + \left(d_1 \beta^2 - \frac{\alpha d_2}{\beta} \right) p + \beta^2 = 0, \tag{3.7}$$

where $p = \dfrac{n^2}{l^2}$. Solving α from (3.7), we obtain that

$$\alpha(p) = d_1 \beta p + \frac{\beta^3}{d_2 p} + \frac{d_1}{d_2} \beta^3. \tag{3.8}$$

We also solve p from the equation (3.7), and we have

$$p = p_\pm(\alpha) = \frac{\alpha d_2 - d_1 \beta^3 \pm \sqrt{(d_1 \beta^3 - \alpha d_2)^2 - 4 d_1 d_2 \beta^4}}{2 d_1 d_2 \beta}. \tag{3.9}$$

Since we require that $\alpha < \beta$, we define

$$\widetilde{l}_{n,+} := \frac{n}{\sqrt{p_+(\beta)}}, \quad \widetilde{l}_{n,-} := \frac{n}{\sqrt{p_-(\beta)}}, \quad n = 1, 2, \ldots, \tag{3.10}$$

where $p_\pm$ are defined in (3.9). For a fixed $n \in \mathbb{N}$, if $\widetilde{l}_{n,+} < l < \widetilde{l}_{n,-}$, then there exists a unique $\alpha_n^S := \alpha(n^2/l^2)$ such that $D_n(\alpha_n^S) = 0$. These points α_n^S are potential steady state bifurcation points.

The function $\alpha(p)$ and the functions $p_\pm(\alpha)$ satisfy the following properties.

Lemma 3.2 *Define*

$$p_* = \frac{\beta}{\sqrt{d_1 d_2}}, \quad \alpha_* = \beta^2 \sqrt{\frac{d_1}{d_2}} \left(2 + \beta \sqrt{\frac{d_1}{d_2}} \right). \tag{3.11}$$

Then the function $\alpha(p) : (0, \infty) \to \mathbb{R}^+$ defined in (3.8) has a unique critical point at p_*, which is the global minimum of $\alpha(p)$ in $(0, \infty)$, and $\lim\limits_{p \to 0^+} \alpha(p) = \lim\limits_{p \to \infty} \alpha(p) = \infty$. Consequently for $\alpha \geq \alpha_* := \alpha(p_*)$, $p_\pm(\alpha)$ are well defined as in (3.9); $p_+(\alpha)$ is strictly increasing and $p_-(\alpha)$ is strictly decreasing, and $p_+(\alpha_*) = p_-(\alpha_*) = p_*$.

Proof Differentiating $D(\alpha(p), p) = 0$ twice and letting $\alpha'(p) = 0$, we obtain that

$$\frac{d}{dp} D(\alpha(p), p) = 2 p d_1 d_2 + d_1 \beta^2 - \alpha'(p) p \frac{d_2}{\beta} + \alpha(p) \frac{d_2}{\beta} = 0,$$

$$\frac{d^2}{dp^2}D(\alpha(p), p) = 2d_1d_2 - \alpha''(p)p\frac{d_2}{\beta} - \alpha'(p)\frac{2d_2}{\beta} = 0,$$

If $\alpha'(p) = 0$, then $\alpha''(p)p\frac{d_2}{\beta} - 2d_1d_2 = 0$, which is

$$\alpha''(p) = \frac{2d_1\beta}{p} > 0.$$

Therefore for any critical point p of $\alpha(p)$, we must have $\alpha''(p) > 0$, and thus the critical point must be unique and it is a local minimum point.

It is easy to see that $\lim_{p\to 0^+} \alpha(p) = \lim_{p\to\infty} \alpha(p) = \infty$, hence the unique critical point p_* is the global minimum point. Since (3.9) is also obtained by (3.7), then $\Lambda = \{(\alpha(p), p) : 0 < p < \infty\}$ and the curve $(\alpha, \alpha_\pm(p))$ are identical. Then the properties of $\alpha(p)$ determine the monotonicity and limiting behavior of $p_\pm(\alpha)$. □

From Lemma 3.2, if for some i, j with $i < j$, we have $\alpha(p_i) = \alpha(p_j)$, then $p_-(\alpha_i^S) = p_+(\alpha_j^S)$ must hold. In this case, for $\alpha = \alpha_i^S = \alpha_j^S$, 0 is not a simple eigenvalue of $L(\alpha)$, and we shall not consider bifurcations at such points. We notice that, from the properties of $p_\pm(\alpha)$ in Lemma 3.2, the multiplicity of 0 as an eigenvalue of $L(\alpha)$ is at most 2. On the other hand, it is also possible that some $\alpha_i^S = \alpha_j^H$ (a Hopf bifurcation point). So the dimension of center manifold of the steady state (u_α, v_α) can be from 1 to 4. Following an argument in [22], we can prove that there are only countably many $l > 0$, in fact only finitely many $l \in (0, M)$ for any given $M > 0$, such that $\alpha = \alpha_i^S = \alpha_j^H$ or $\alpha_i^S = \alpha_j^S$ for these l and $i, j \in \mathbb{N}$. We define the set of these l to be

$$L^E = \{l > 0 : \alpha(i^2/l^2) = \alpha(j^2/l^2) \text{ or } \alpha(i^2/l^2) = \tilde{\alpha}(j^2/l^2),\ \alpha \in [\alpha_*, \infty),\ i, j \in \mathbb{N}\}, \tag{3.12}$$

where $\alpha(p)$ is the function defined in (3.8), and

$$\tilde{\alpha}(p) = \beta^3 + \beta(d_1 + d_2)p, \tag{3.13}$$

following (2.12). Then the points in L^E can be arranged as a sequence whose only limit point is ∞. Finally we show that $\frac{d}{d\alpha}D_j(\alpha_j^S) \neq 0$. By direct calculation, we have

$$\frac{d}{d\alpha}D_j(\alpha_j^S) = -p_j\frac{d_2}{\beta} < 0, \quad p_j = \frac{j^2}{l^2}.$$

Now we state the main result about the steady state solution bifurcation of (3.1).

Theorem 3.3 *Assume that*

$$0 < \beta < (\sqrt{2} - 1)\sqrt{\frac{d_2}{d_1}}, \tag{3.14}$$

and let $n \in \mathbb{N}$. Suppose that $l \in (\widetilde{l}_{n,+}, \widetilde{l}_{n,-}) \setminus L^E$, where $\widetilde{l}_{n,\pm}$ is defined in (3.10) and L^E is a countable subset of $\mathbb{R}^+$ defined in (3.12). Then $\alpha_n^S = \alpha(\frac{n^2}{l^2})$ satisfies $\alpha_* < \alpha_n^S < \beta$, and $\alpha = \alpha_n^S$ is a bifurcation point for (3.1). Moreover,

1. *There exists a smooth curve $\Gamma_n \in C^\infty$ of positive solutions of (3.1) bifurcating from $(\alpha, u, v) = (\alpha_n^S, u_{\alpha_n^S}, v_{\alpha_n^S})$, with Γ_n contained in a global branch C_n of positive solutions of (3.1).*
2. *Near $(\alpha, u, v) = (\alpha_n^S, u_{\alpha_n^S}, v_{\alpha_n^S})$, $\Gamma_n = \{(\alpha_n(s), u_n(s), v_n(s)) : s \in (-\varepsilon, \varepsilon)\}$, where $u_n(s) = \beta + s a_n \cos(nx/l) + s\psi_{1,n}(s)$, $v_n(s) = \dfrac{\beta + \alpha_n^S}{2\beta^2} + s b_n \cos(nx/l) + s\psi_{2,n}(s)$, for $s \in (-\varepsilon, \varepsilon)$ for some C^∞ smooth functions $\alpha_n, \psi_{1,n}, \psi_{2,n}$ such that $\alpha_n(0) = \alpha_n^S$, $\psi_{1,n}(0) = \psi_{2,n}(0) = 0$; Here a_n, b_n satisfy*
$$L(\alpha_n^S)[(a_n, b_n)^T \cos(nx/l)] = (0, 0)^T. \tag{3.15}$$
3. *Either C_n contains another $(\alpha, u, v) = (\alpha_m^S, u_{\alpha_m^S}, v_{\alpha_m^S})$ for $m \neq n$, or the projection $\mathbf{proj}_\alpha C_n$ of C_n on the α-axis satisfies*
$$(-\beta + \delta, \beta) \supset \mathbf{proj}_\alpha C_n \supset (\alpha_n^S, \beta), \tag{3.16}$$
for some $\delta > 0$.

Proof If β satisfies (3.14), then $\alpha_* < \alpha_n^S < \beta$. Since we have verified conditions (H_2) in previous paragraphs, then a bifurcation of steady state solutions occurs at $\alpha = \alpha_n^S$. Note that we exclude L^E (as defined in (3.12)), so $\alpha = \alpha_j^S$ is always a bifurcation from a simple eigenvalue point. From the global bifurcation theorem in Shi and Wang [14], Γ_n is contained in a global branch C_n of solutions. Hence the results stated here are all proved except (i) C_n consists of positive solutions only; and (ii) part 3.

Firstly we prove that any solution on C_n is positive if $\alpha \in (-\beta, \beta)$. This is true for solutions on Γ_n as $\beta > 0$ and $\alpha_n^S + \beta > 0$. Suppose that there is a solution on C_n which is not positive. Then by the continuity of C_n, there exists an $(\alpha_\sharp, u_\sharp, v_\sharp) \in C_n$ such that $\alpha_\sharp \in (-\beta, \beta)$, $u_\sharp(x) \geq 0$, $v_\sharp(x) \geq 0$ for $x \in \overline{\Omega}$, and there exists $x_0 \in \overline{\Omega}$, such that $u_\sharp(x_0) = 0$ or $v_\sharp(x_0) = 0$. Suppose that $u_\sharp(x_0) = 0$, then $u_\sharp$ reaches its minimum at x_0. If $x_0 \in \Omega$, then we get a contradiction with $-d_1 \Delta u_\sharp(x_0) = (\beta - \alpha)/2 > 0$. If $x_0 \in \partial\Omega$, then near $x = x_0$, we have $-d_1 \Delta u_\sharp(x_0) > 0$ and x_0 is the local minimum, then $\partial_n u_\sharp(x_0) < 0$ which contradicts with the zero Neumann boundary condition. Thus $u_\sharp(x_0) = 0$ is impossible. Similarly we can prove that $v_\sharp(x_0) = 0$ is impossible. This shows that any solution of (3.1) on C_n is positive as long as $\alpha \in (-\beta, \beta)$.

Secondly we notice that when $\alpha = -\beta$, the only solution of (3.1) is $(u, v) = ((\beta - \alpha)/2, 0)$. And as we have shown here the only bifurcation points for steady state solutions are $\alpha = \alpha_n^S$. Thus $\mathbf{proj}_\alpha C_n$ must have a lower bound which is strictly larger than $-\beta$. From Lemma 3.1, all positive solutions of (3.1) are uniformly bounded for $\alpha \in [-\beta, \beta - \delta)$. Hence C_n cannot be unbounded for $\alpha \in [-\beta, \beta - \delta)$. From the global bifurcation theorem in [14], either C_n contains another $(\alpha, u, v) = (\alpha_m^S, u_{\alpha_m^S}, v_{\alpha_m^S})$ for $m \neq n$, or C_n is unbounded, or C_n intersects the boundary of $(-\beta, \beta) \times X \times X$,

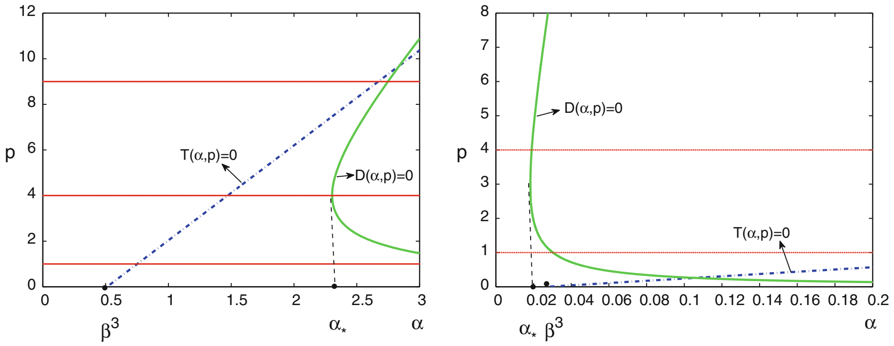

Fig. 3 Graph of $T(\alpha, p) = 0$ and $D(\alpha, p) = 0$. (*Left*) $\beta = 0.8$, $d_1 = 0.2$, $d_2 = 0.1$ and $l = 1$; (*Right*) $\beta = 0.3$, $d_1 = 0.01$, $d_2 = 1$ and $l = 3$. The *horizontal lines* are $p = n^2/l^2$

where X is a function space. But either one in the latter two alternatives implies that $\mathbf{proj}_\alpha C_n \supset (\alpha_n^S, \beta)$. This completes the proof. □

Remark 3.4 1. The continuum C_n in Theorem 3.3 is possibly unbounded as $\alpha \to \beta^-$ as the upper bounds in Lemma 3.1 tend to infinity as $\alpha \to \beta^-$. But it is also possible that C_n can be extended to $\alpha = \beta$, which corresponds to the case $a = 0$ for the original parameters. Since $a \le 0$ is not physically reasonable, we do not discuss the case that $a \le 0$.

2. If $d_1 \ge d_2$, then all the bifurcating steady state solutions on Γ_n near $\alpha = \alpha_n^S$ are unstable since $\alpha_n^S > \alpha_* > \beta^3\sqrt{d_1/d_2} \ge \beta^3 = \alpha_0^H$, which is the primary Hopf bifurcation point.

4 Numerical simulations and discussion

We have identified two critical parameter values for the system (1.2): $\alpha = \beta^3$, which is the smallest Hopf bifurcation point, and $\alpha = \alpha_*$ (defined as in (3.11)). The constant steady state (u_α, v_α) is locally asymptotically stable when $\alpha < \min\{\beta^3, \alpha_*\}$. When $\beta^3 < \alpha_*$, then (u_α, v_α) loses the stability at $\alpha = \beta^3$ through a Hopf bifurcation; when $\alpha_* < \beta^3$, a steady state bifurcation is likely to happen for some $\alpha \in (\alpha_*, \beta)$. The left panel of Fig. 3 shows the graph of $T(\alpha, p) = 0$ and $D(\alpha, p) = 0$ with a set of parameters so that $\beta^3 < \alpha_*$, while the right panel shows the one for the case $\alpha_* < \beta^3$.

For the latter case, one can choose the length parameter l so that there are steady state bifurcation points in (α_*, β). Indeed for the parameters given in the right panel of Fig. 3, if we choose $l = 3$, then we have

$$\begin{aligned} &\alpha_5^S = 0.0183 < \alpha_6^S = 0.019 < \alpha_4^S = 0.0208 < \alpha_7^S = 0.0216 \\ &< \alpha_8^S = 0.0254 < \alpha_0^H = 0.027 < \alpha_3^S = \alpha_9^S = 0.0303. \end{aligned} \tag{4.1}$$

We use several numerical simulations to illustrate and complement our analytical results. The simulations are generated by using the `Matlab` pdepe solver which solves initial-boundary value problems for parabolic-elliptic PDEs in 1-D. For the parameters

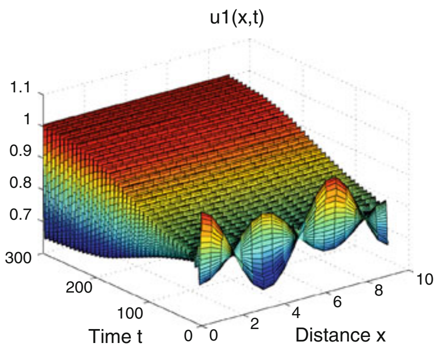

Fig. 4 Numerical simulation of the system (1.2). (*Left*) $u(x, t)$; (*Right*) $v(x, t)$. Here $\alpha = 0.52, \beta = 0.8, d_1 = 0.2, d_2 = 0.1, l = 3, 0 \leq t \leq 300$, and the initial values $u_0(x) = 0.8 + 0.1 \cos(x)$; $v_0(x) = 1.03 + 0.1 \cos(x)$. The solution converges to a spatially homogenous periodic orbit

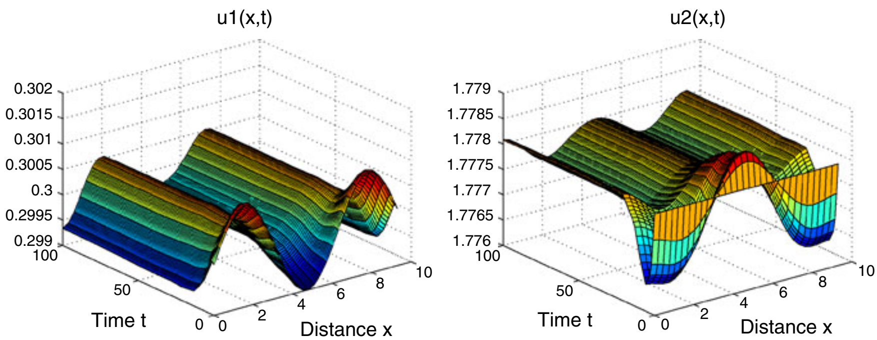

Fig. 5 Numerical simulation of the system (1.2). (*Left*) $u(x, t)$; (*Right*) $v(x, t)$. Here $\alpha = 0.02, \beta = 0.3, d_1 = 0.01, d_2 = 1, l = 3, 0 \leq t \leq 100$, and the initial values $u_0(x) = 0.3 + 0.001 \sin(x)$; $v_0(x) = 1.778$. The solution converges to a spatially non-homogenous steady state solution

given in the left panel of Fig. 3 with $l = 3$, a simulation with $\alpha = 0.52 > \alpha_0^H = 0.512$ is shown in Fig. 4, which is dominated by the ODE limit cycle dynamics. On the other hand, for the parameters given in the right panel of Fig. 3 with $l = 3$, several steady state bifurcations occur at α-values smaller than α_0^H (see (4.1)). Figures 5, 6 and 7 show three different solution trajectories with different initial conditions. While each shows a stationary spatial pattern, the one in Fig. 5 appears to have spatial period $3\pi/2$, which corresponds to $n = 4$ (mode $\cos(4x/3)$); the one in Fig. 6 appears to have spatial period π, which corresponds to $n = 6$ (mode $\cos(2x)$); and finally the one in Fig. 7 is not spatially periodic but with peaks of different height. For the simulations in Figs. 6 and 7, if the integration time is chosen longer, then the observed patterns can switch to a different pattern. This suggests the spatial patterns observed in earlier time may be unstable steady states. Note that such asymmetric peak solutions have also been observed in [17] for (1.2). From (4.1), the parameter $\alpha = 0.02$ in Figs. 5, 6 and 7 is between $\alpha_6^S = 0.019$ and $\alpha_4^S = 0.0208$.

Our analytical results given here and the numerical simulations guided by the analytical results show a rough picture of the dynamics in terms of system parameters.

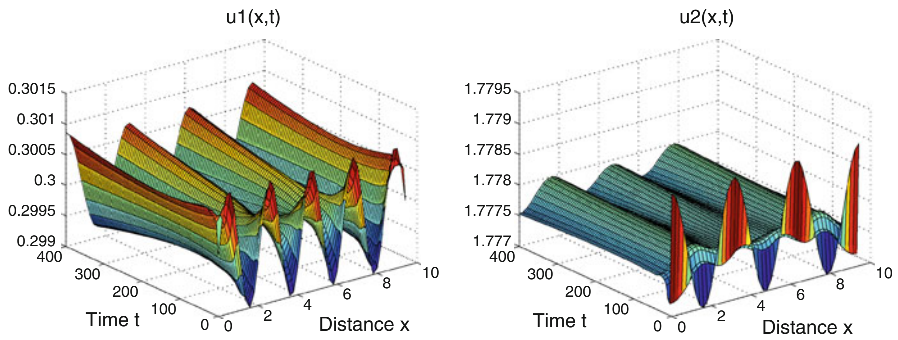

Fig. 6 Numerical simulation of the system (1.2). (*Left*) $u(x, t)$; (*Right*) $v(x, t)$. Here $\alpha = 0.02, \beta = 0.3, d_1 = 0.01, d_2 = 1, l = 3, 0 \le t \le 400$, and the initial values $u_0(x) = 0.3 + 0.001\sin(3x)$; $v_0(x) = 1.778 + 0.001\cos(2x)$. The solution shows a spatially periodic pattern with wave length π

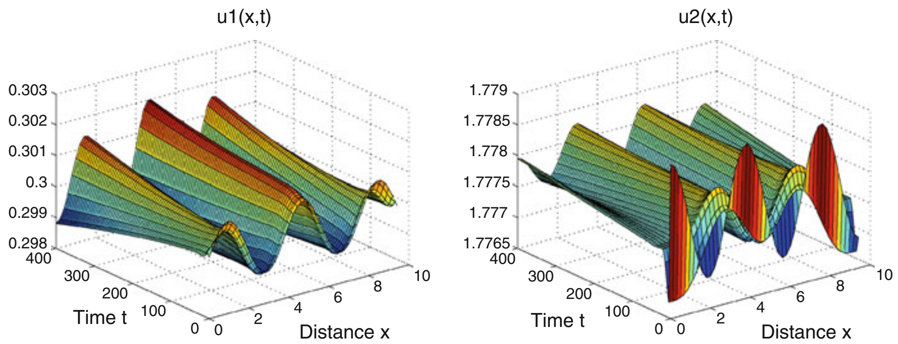

Fig. 7 Numerical simulation of the system (1.2). (*Left*) $u(x, t)$; (*Right*) $v(x, t)$. Here $\alpha = 0.02, \beta = 0.3, d_1 = 0.01, d_2 = 1, l = 3, 0 \le t \le 400$, and the initial values $u_0(x) = 0.3+0.001\sin(5x/3)$; $v_0(x) = 1.778 + 0.001\cos(5x/3)$. The solution shows a spatially asymmetric pattern

There are several parameter regimes where the dynamical behavior of (1.2) are drastically different.

1. When β is large, then for any $\alpha \in (-\beta, \beta), d_1, d_2 > 0$, the solution of (1.2) appears to converge to the constant steady state. Indeed our analytical results show that when $0 < \beta < \min\{\beta^3, \alpha_*\}$, ($\alpha_*$ is defined in (3.11)), that is equivalent to

$$\beta > \max\left\{1, (\sqrt{2} - 1)\sqrt{\frac{d_2}{d_1}}\right\}, \tag{4.2}$$

then (u_α, v_α) is locally asymptotically stable for any $\alpha \in (-\beta, \beta)$. The convergence to the constant steady state can be easily seen by simulation, but the analytical proof of global stability for PDE remains open. We also conjecture that (u_α, v_α) is globally asymptotically stable for any $\alpha \in (-\beta, 0]$ regardless the value of β (which corresponds to the case $b < a$ in terms of original parameters.)

2. When $0 < \beta^3 < \beta < \alpha_*$ is satisfied, that is

$$(\sqrt{2}-1)\sqrt{\frac{d_2}{d_1}} < \beta < 1, \tag{4.3}$$

then there are a sequence of Hopf bifurcation points $\beta^3 = \alpha_0^H < \alpha_1^H < \cdots < \alpha_n^S < \beta$ where periodic orbits of (1.2) bifurcate out from the constant steady state (u_α, v_α) (see Theorem 2.1). In particular, (u_α, v_α) loses the local stability to a spatially homogenous periodic orbit at $\alpha = \beta^3$. On the other hand, since $\beta < \alpha_*$, there is no steady state bifurcation for any $\alpha \in (-\beta, \beta)$. Hence the parameter regime given by (4.3) is dominated by time-periodic patterns but probably not stationary spatially nonhomogeneous patterns. The number n of spatially nonhomogeneous Hopf bifurcation points depends on the length parameter l.

3. When $0 < \alpha_* < \beta < \beta^3$ is satisfied, that is

$$1 < \beta < (\sqrt{2}-1)\sqrt{\frac{d_2}{d_1}}, \tag{4.4}$$

then there exist a number of steady state solution bifurcation points α_j^S for $m \le j \le n$ and $m, n \in \mathbb{N}$ where spatially nonhomogeneous steady state solutions bifurcate from the constant steady state (u_α, v_α) (see Theorem 3.3). Again the number $n - m$ of spatially nonhomogeneous Hopf bifurcation points depends on the length parameter l. On the other hand, since $\beta < \beta^3$, there is no Hopf bifurcation for any $\alpha \in (-\beta, \beta)$. Hence the parameter regime given by (4.4) is dominated by stationary spatially nonhomogeneous patterns but probably not time-periodic patterns.

4. Finally if

$$0 < \beta < \min\left\{1, (\sqrt{2}-1)\sqrt{\frac{d_2}{d_1}}\right\}, \tag{4.5}$$

then $0 < \max\{\alpha_*, \beta^3\} < \beta$. In this case both the results in Theorem 2.1 and the ones in Theorem 3.3 are applicable. Hence possibly both Hopf bifurcations and steady state bifurcations occur for $\alpha \in (\min\{\alpha_*, \beta^3\}, \beta)$, and these bifurcation points form an intertwining sequence of bifurcation points. The pattern formation in the parameter regime (4.5) is perhaps the most complicated one.

We also comment that near the codimension-two Hopf–Turing bifurcation point, where the two instabilities happen at the same parameters, these instabilities can compete with each other which results in more complicated spatiotemporal patterns. The Hopf–Turing point in this model is the intersection point of curves $T(\alpha, p) = 0$ and $D(\alpha, p) = 0$ in Fig. 3. Some previous studies in Hopf–Turing bifurcations can be found in [1,7,10].

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